Last time:

K complete, non-arch. valued, $|\cdot|: K \to \mathbb{R}_{\geq 0}$

L/K algebraic $\quad$ [e.g. $L = \overline{K}$)

$\Rightarrow \exists !$ ext. $|\cdot|_L : L \to \mathbb{R}_{\geq 0}$ of $|\cdot|$

If L/K finite $\Rightarrow$ L complete

$$\& \quad |x|_L = |N_{L/K}(x)|^{\frac{1}{n}}$$

$$n = [L:K]$$

If K is discretely valued & L/K finite

$\Rightarrow$ L is discretely valued

& $\mathcal{O}_L$ free over $\mathcal{O}_K$ of rk $n = [L:K]$

Need to finish (assuming K discr. valued):

Lemma: $\pi \in \mathcal{O}_K$ uniformizer

$f: M \to N$ morphism of $\mathcal{O}_K$-modules

1) If M is π-adically complete,

(i.e. $M \simeq \varprojlim_n M/\pi^n M$)

N is π-adically sep. (i.e. $\bigcap \pi^n N = \{0\}$)

& $\bar{f}: M/\pi M \to N/\pi N$ surj

$\Rightarrow$ f is surj.

2) If M is π-adic. sep, N is π-tors. free

& $\bar{f}: M/\pi M \to N/\pi N$ is inj.

$\Rightarrow$ f is injective

Pf: 1) Let $n \in N$ and let $m_1 \in M$,

s.t. $\bar{f}(\overline{m_1}) = \bar{n}$

$\Rightarrow$ $f(m_1) - n = \pi \cdot n_1$, $n_1 \in N$

Let $m_2 \in M$, s.t. $\bar{f}(\overline{m_2}) = -\overline{n_1}$

$\Rightarrow$ $f(m_1 + \pi \cdot m_2) = n + \pi n_1 + \pi(-n_1 + \pi n_2)$

$= n + \pi^2 \cdot n_2$ for some $n_2 \in N$,

s.t. $f(m_2) = -n_1 + \pi n_2$

$\Rightarrow$ continue

$\Rightarrow$ Find $m_1, \dots, m_\ell \in M$, s.t.

$$f\left(\sum_{i=0}^{\ell-1} \pi^i \cdot m_{i+1}\right) - n \in \pi^{\ell+1} N$$

Set $m := \sum_{i=0}^{\infty} \pi^i \cdot m_{i+1} \in M$ (exists by completeness)

$\Rightarrow f(m) - n \in \bigcap_{\ell \geq 0} \pi^\ell \cdot N \underset{N \text{ sep.}}{=} \{0\}$

$\Rightarrow f(m) = n$ as desired

2) Let $m \in \ker f$

Know: $\bar{f} : M/_\pi \to N/_\pi$ inj.

$\Rightarrow \quad m \in \pi \cdot M$, i.e. $m = \pi \cdot m_1$, $m_1 \in M$

$\Rightarrow 0 = f(m) = \pi \cdot f(m_1)$

$\Rightarrow \quad f(m_1) = 0 \quad \Rightarrow \quad m_1 = \pi \cdot m_2$

N π-tors. free

$\Rightarrow \dots\dots \Rightarrow m = \pi m_1 = \pi^2 \cdot m_2 = \dots$

$$\in \bigcap_{n \geq 0} \pi^n M \underset{\substack{\uparrow \\ M\ \pi\text{-sep.}}}{=} \{0\}$$

Ex: $M = K$, $N = 0$, $K = \mathrm{Frac}(\mathcal{O}_K)$

$\Rightarrow \quad M/\pi M = 0$ as $M = \pi \cdot M$

but $M \neq 0$

$\Rightarrow$ Assumptions are necessary

Proof of "$\mathcal{O}_L$ free over $\mathcal{O}_K$" apply this last lemma to

$$\varphi : \mathcal{O}_K^m \to \mathcal{O}_L \quad \text{which is an iso.}$$

mod π $\qquad \xi$

π-adically complete,
π-torsionfree

Newton polygons

K discretely valued, complete

$v : K \to \mathbb{Z} \cup \{\infty\}$ normalized add. valuation

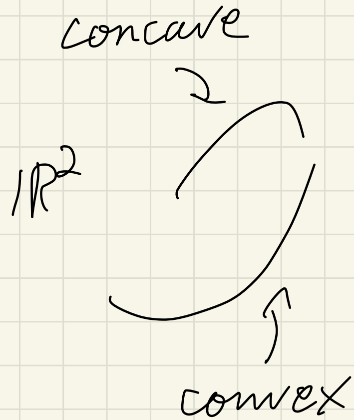
Let $f(x) = a_n x^n + a_{n-1} \cdot x^{n-1} + \dots + a_0 \in K[x]$

Define the Newtonpolygon of f as the lower convex envelope of

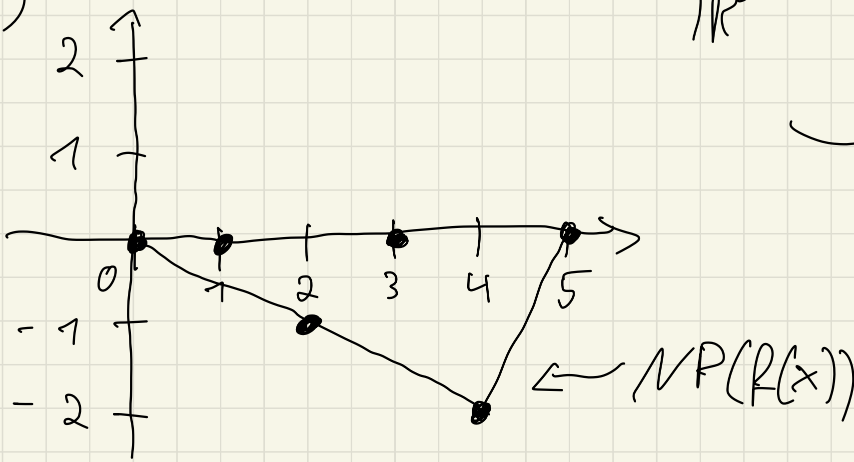
$$\{(i, v(a_i))\} \subseteq \mathbb{R}^2$$

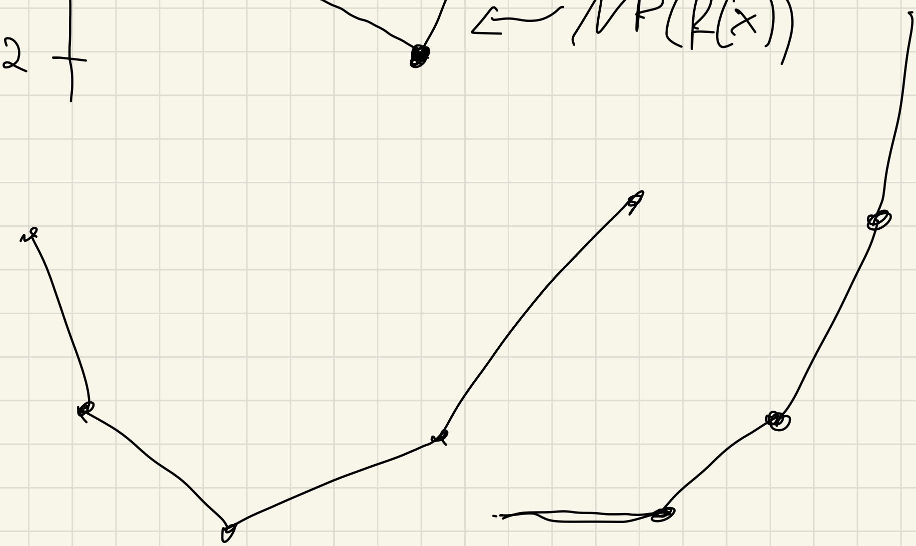
Notation: $NP(f(x))$

E.g.: 1) $f(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5$
$\in \mathbb{Q}_2[x]$

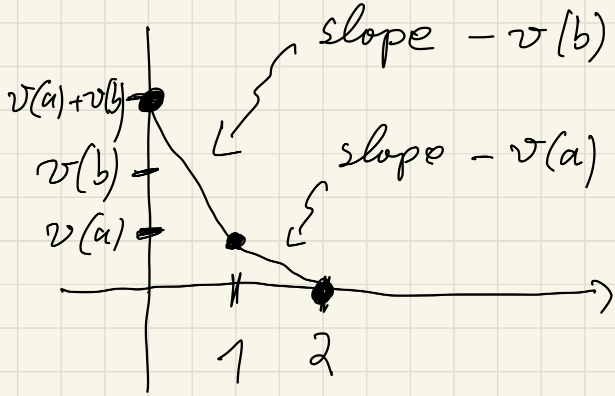


$\Rightarrow$





2) $f(x) = (x-a)(x-b)$, $a, b \in \mathcal{O}_K$
$= x^2 - (a+b)x + a \cdot b$



Assume $v(a) < v(b)$

$\Rightarrow v(a+b) = \min(v(a), v(b))$

$= v(a)$

$v(a) + v(b) = v(a \cdot b)$

$\Rightarrow$ $NP(f(x))$ "knows" $v(a), v(b)$

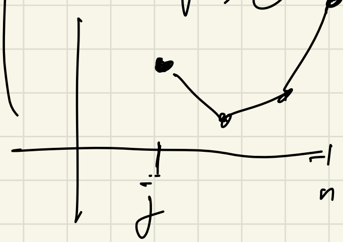
Note: a, b are the roots of f

$f(x) = a_j x^j + \dots + a_n x^n$

$j > 0$

$a_j = 0$

$a_n =$



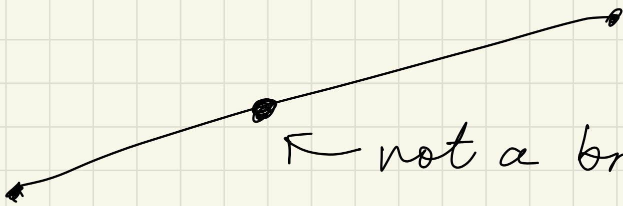
In general, let

$(q_0, t_0), (q_1, t_1), \dots, (q_r, t_r) \in \mathbb{Z}^2$

be the break points of $NP(f(x))$

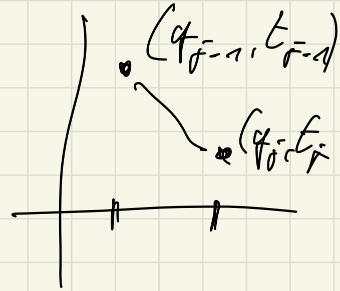
(e.g. if $a_0 \neq 0 \rightsquigarrow (q_0, t_0) = (0, v(a_0))$

if $a_n \neq 0 \rightsquigarrow (q_r, t_r) = (n, v(a_n))$



← not a breakpoint)

& $s_j := \dfrac{t_{j-1} - t_j}{q_j - q_{j-1}}$



the <u>negative</u> slope

with multiplicity $m_j = q_j - q_{j-1}$

Assume $a_0 \neq 0,\ a_n \neq 0$

Prop: $f(X)$ has exactly m_j roots in $\overline{K}$ with valuation s_j (counted with multiplicities)

(unique ext. of v to $v: \overline{K} \to \mathbb{Q} \cup \{\infty\}$)

Proof: $NP(\lambda \cdot f(x)) = NP(f(x)) + (0, v(\lambda))$

$\lambda \in K^\times$

$\Rightarrow$ Wlog $a_0 = 1$

$\Rightarrow$ $f(x) = (1-\alpha_1 x)\cdot(1-\alpha_2 x)\cdot\ldots\cdot(1-\alpha_n \cdot x)$

$\alpha_1, \ldots, \alpha_n \in \overline{K}$

(i.e. $x^n f(\frac{1}{x}) = (x-\alpha_1)\cdot\ldots\cdot(x-\alpha_n)$)

Let $\varrho_1 < \ldots < \varrho_{r'}$ be the dist. valuations of the α_j's

$m_j' = \#\{1 \leq i \leq n \mid v(\alpha_i) = \varrho_j\}$

$j = 1, \ldots, r'$

Label $\alpha_1, \ldots, \alpha_n$, s.t.

$\varrho_1 = v(\alpha_1) = \ldots = v(\alpha_{m_1'})$

$q_0' = 0$ $\qquad\qquad\qquad\qquad q_1' = m_1'$

$< \varrho_{2'} = v(\alpha_{m_1'+1}) = \dots = v(\alpha_{\underbrace{m_1'+m_2'}_{q_2'}})$

$< \dots < \varrho_{r'} = v(\alpha_{q_{r'-1}'}) = \dots \; v(\alpha_{\underbrace{q_{r'-1}'+m_{r'}'}_{q_{r'}'=n}})$

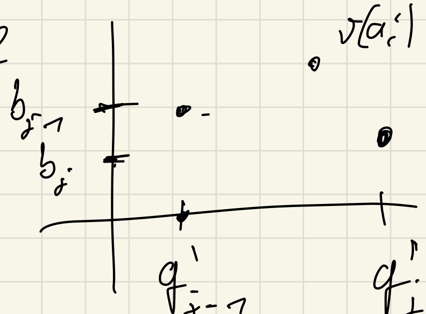
Need to see: $r = r'$, $\varrho_j = -s_j$, $m_j' = m_j$

Note: $a_i = (-1)^i \sum\limits_{1 \le j_1 < \dots < j_i \le n} \alpha_{j_1} \cdot \dots \cdot \alpha_{j_n}$

$\Rightarrow v(a_{q_j'}) = v(\alpha_1 \cdot \dots \cdot \alpha_{q_j'})$ (all other summands have strictly larger val.)

strict Δ-ineq.

$$b_j = \sum_{\ell=1}^{j} \varrho_\ell \cdot m_\ell'$$



while for

$q_{j-1}' < i \le q_j'$

$$\Rightarrow v(a_i) \geq \underbrace{\sum_{l=1}^{j-1} \varrho_l \cdot m_j'}_{= b_{j-1}} + (i - q_{j-1}') \cdot \varrho_j$$

$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{v(\alpha_1 \cdot \ldots \cdot \alpha_i)}}$$

as a fct. of i this defines the line from

(q_{j-1}', b_{j-1}) to $(q_j' \; b_j)$

$\Rightarrow$ $NP(f(x))$ has break points (note $\varrho_{j-1} < \varrho_j$)
$(q_j' \, . \, b_j)$ with slopes ϱ_j

as $f(x)$ has roots α_j^{-1}

$\Rightarrow \quad s_j = -\varrho_j$ $\square$

Corollary: 1) If $f(x) \in K[x]$ irreducible
$\Rightarrow$ $NP(f(x))$ has exactly one slope

2) If $NP(f(x))$ has one slope $\mu = \frac{r}{n}$,
$\gcd(r,n) = 1$ & $n = \deg f$

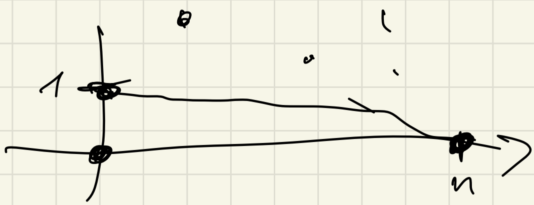
$\Rightarrow$ f irreducible

(In part if f is Eisenstein:

$$x^n + \pi a_{n-1} x^{n-1} + \dots + \pi \cdot a_1 \cdot x^1 + a_0 \cdot \pi$$

with $a_{n-1}, \dots, a_1 \in \mathcal{O}_K$, $a_0 \in \mathcal{O}_K^\times$

$\Rightarrow$



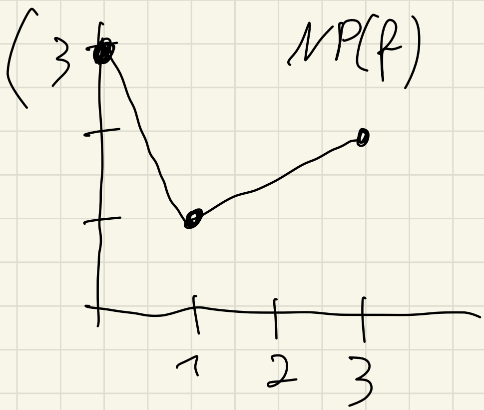
)

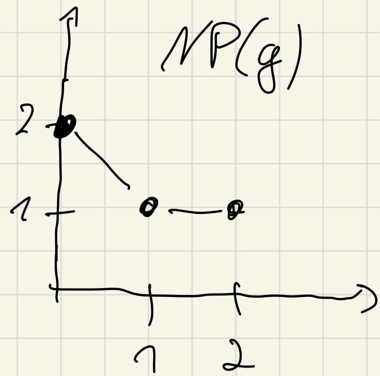
Proof: 1) ✓ as Galois action preserves valuation

2) Last prop. implies the following
If $f, g \in K[x]$

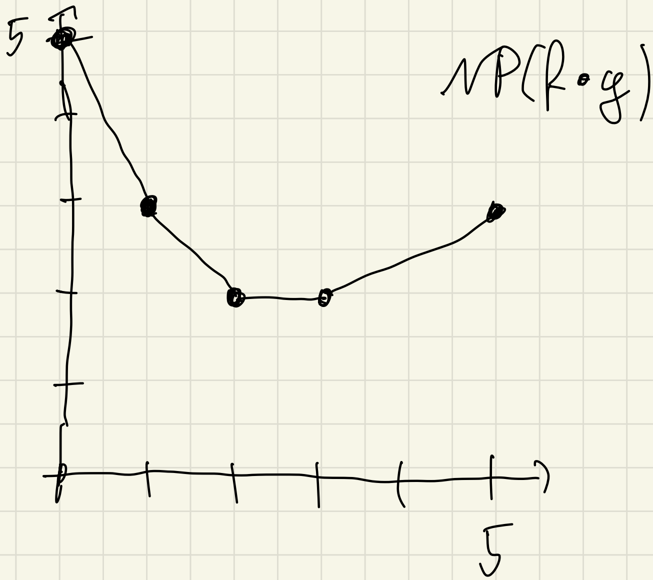
$\Rightarrow$ $NP(f \cdot g)$ is obtained from
$NP(f)$, $NP(g)$

by "concatenation of slopes"





conc. $\rightsquigarrow$



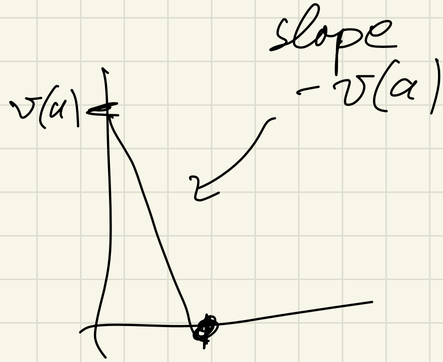
$\Rightarrow$ If f not irreducible, then $NP(f)$ has integral points different from the two ends

But $(r,n) = (1)$, $n = \deg f$

$\Rightarrow$ there exist no such integral points

$\square$

$$NP(X-a)$$



$$NP(f\cdot g) = NP(f) * NP(g)$$

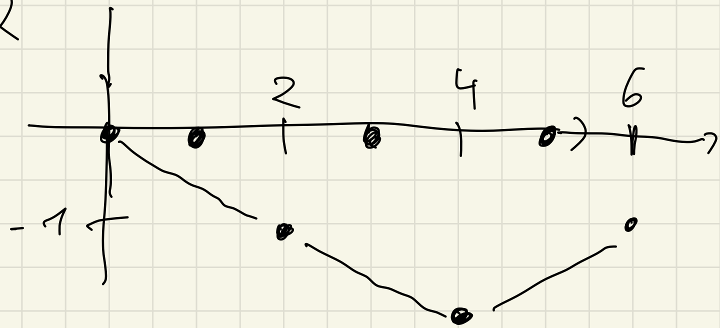
concatenation of slopes

Ex: Consider $f(x) =$
$$1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5 + \frac{1}{6}x^6$$
$\in \mathbb{Q}[X]$

Claim: f is irred. in $\mathbb{Q}[X]$

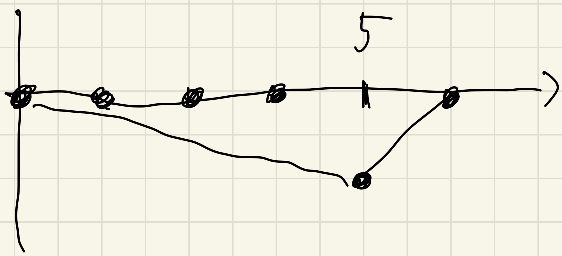
Prf: Draw $NP_p(f)$ for $f \in \mathbb{Q}_p[X]$ for diff. primes p

$p = 2$



$\Rightarrow f$ has no root in $\mathbb{Q}_2$
(as no slope is integral
$-\frac{1}{2}, \frac{1}{2}$)

$p = 5$



$\Rightarrow \quad f = f_1 \cdot f_2 \in \mathbb{Q}_5[x]$,
$\deg f_1 = 1$, $\deg f_2 = 5$

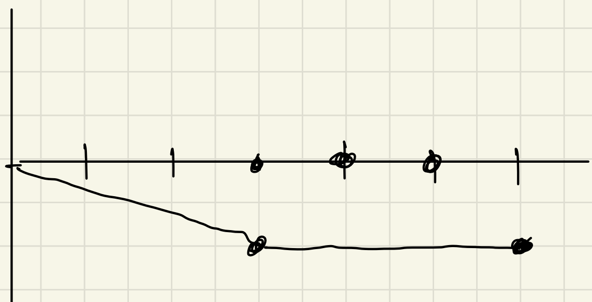
& f_2 is irred (by last corollary)

If $f \in \mathbb{Q}[x]$ was not irred.

$\Rightarrow$ f must have a linear factor $\lightning$

(p = 2)

$p = 3$



$$\Rightarrow f = f_1 \cdot \underbrace{f_2}_{} \quad \text{with } \deg f_1 = 3, \; \deg f_2 = 3$$

$\in \mathbb{Q}_3[x]$

f_1 irred & $NP(f_1)$

($f_1 = (x-\alpha_1)\cdot(x-\alpha_2)(x-\alpha_3) \in \mathbb{Q}_3$

$\alpha_1, \ldots, \alpha_3 \in \overline{\mathbb{Q}_3}$ the roots with

$-v(\alpha_1) = -\frac{1}{3}$)